Contemporary Mathematics

$H_\infty \neq E_\infty$

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Abstract. We provide an example of a spectrum over S^0 with an H_∞ structure which does not rigidify to an E_3 structure. It follows that in the category of spectra over S^0 not every H_∞ ring spectrum comes from an underlying E_∞ ring spectrum. After comparing definitions, we obtain this example by applying Σ^∞_+ to the counterexample to the transfer conjecture constructed by Kraines and Lada.

1. Introduction

In recent years there has been a renewed interest in the study of E_∞ ring spectra and their strictly commutative analogues, commutative S-algebras. Such spectra are equipped with a well-behaved theory of power operations. This structure provides formidable computational tools which can be used to deduce a number of surprising results (for some examples see [**1**, Ch. 2]).

Such operations determine and are determined by an H_∞ ring structure, the analogue of an E_∞ ring structure in the stable *homotopy* category. The theory of power operations is sufficiently rich that one might conjecture that every H_∞ ring spectrum is obtained by taking an E_∞ ring spectrum and then passing to the homotopy category.

This turns out to be a stable analogue of the transfer conjecture, a conjectural equivalence between the homotopy category of infinite loop spaces and a subcategory of the homotopy category of based spaces whose objects admit certain transfer homomorphisms (see [**4**] for a more complete description).

Kraines and Lada demonstrate the falsehood of the transfer conjecture by contructing an explicit counterexample. In their paper, Kraines and Lada define the notion of an $L(n)$ space. When $n = 2$, this is a space equipped with transfer

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homomorphisms. They make use of the following implications

$$\begin{aligned} X \text{ is an infinite loop space} &\implies X \text{ is an } E_\infty \text{ space} \\ &\iff X \text{ is an } L(\infty) \text{ space} \\ &\implies X \text{ is an } L(n) \text{ space} \\ &\implies X \text{ is an } L(n-1) \text{ space} \ldots \end{aligned}$$

We can view lifting an $L(n)$ structure to an $L(n+1)$ structure and so on as constructing an action of the E_∞ operad up to increasingly coherent homotopy.

Theorem 1.1 ([**4**]). *Let s be a generator of* $\mathrm{Prim} H^{30}(BU; \mathbb{Z}_{(2)})$. *Define KL by the following fibration sequence:*

$$KL \xrightarrow{i} BU_{(2)} \xrightarrow{4s} K(\mathbb{Z}_{(2)}, 30).$$

Then i is a map of $L(2)$ spaces, but the $L(2)$ structure on KL does not lift to an E_3 structure. In particular, KL does not admit an E_∞ structure compatible with this $L(2)$ structure.

After some translation we will prove the following theorem, which provides an example in the category of H_∞ ring spectra augmented over S^0 whose H_∞ structure does not arise by forgetting an E_∞ structure.

Theorem 1.2. *The map*

$$\Sigma^\infty_+ KL \xrightarrow{\Sigma^\infty_+ i} \Sigma^\infty_+ BU_{(2)}$$

is a map of H_∞ ring spectra augmented over S^0, but the H_∞ ring structure on $\Sigma^\infty_+ KL$ does not lift to an E_3 structure. In particular, $\Sigma^\infty_+ KL$ does not admit a compatible E_∞ ring structure.

To prove this we will show that Σ^∞_+ takes $L(2)$ spaces to H_∞ ring spectra under S^0 and takes E_∞ spaces spaces to E_∞ ring spectra under S^0. This comparison is deduced immediately from some of the results in [**6**].

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2. $L(n)$ spaces and spectra

Let $\mathcal{L}$ be the linear isometries operad. We will abuse notation and let L denote the associated reduced monad on pointed spaces with Cartesian products, spaces under S^0 with smash products, and spectra under S^0 with smash products.

In particular:

- L is an endofunctor on pointed spaces satisfying
$$LY = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} Y^n/(\sim),$$
where $\sim$ represents the obvious base point identifications.
- L is an endofunctor on spaces under S^0 satisfying
$$LY = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} Y^n/(\sim),$$
where $\sim$ represents the obvious unit map identifications.

- L is an endofunctor on the Lewis-May-Steinberger category of spectra (see [**5**]) under S^0 satisfying
$$LE = \bigvee_{n\geq 0} \mathcal{L}(n) \ltimes_{\Sigma_n} E^{\wedge n}/(\sim),$$
where $\sim$ represents the obvious unit map identifications (see [**3**, 4.9,6.1]).

We justify this abuse of notation with the following lemma:

Lemma 2.1 ([**6**, 4.8, p. 1027]). *We have the following chain of isomorphisms natural in based spaces*[1] X
$$\begin{aligned}\Sigma^\infty_+ LX &\equiv \Sigma^\infty (LX)_+\\ &\cong \Sigma^\infty L(X_+)\\ &\cong L\Sigma^\infty X_+\\ &\equiv L\Sigma^\infty_+ X.\end{aligned}$$

For simplicity, for the remainder of this paper we will assume all spaces are nondegenerately based and let $e\colon \mathrm{Id} \to L$ and $\mu\colon L^2 = LL \to L$ denote the structure maps of L.

Recall that the category of L-algebras in group-like pointed spaces is equivalent to the category of infinite loop spaces. The following definition provides a categorical filtration between spaces and homotopy coherent L-algebras (which are weakly equivalent to L-algebras).

Definition 2.2. A based space X is $L(n)$ if one can construct maps
$$f_k\colon I^k \times L^{k+1}X \to X \text{ for } k < n$$
such that

(1) the composite $X \xrightarrow{e} LX \xrightarrow{f_0} X$ is the identity,
(2) if $t_j = 0$, $f_k(t_1, \dots, t_k, z) = f_{k-1}\circ(\mathrm{Id}_{I^{k-1}} \times L^{j-1}\mu L^{k-j})(t_1, \dots, \widehat{t_j}, \dots, t_k, z)$,
(3) and if $t_j = 1$, $f_k(t_1, \dots, t_k, z) = f_{j-1}\circ(\mathrm{Id}_{I^{j-1}} \times L^j f_{k-j})(t_1, \dots, \widehat{t_j}, \dots, t_k, z)$.

Remark 2.3. Despite the similarity in notation, we remind the reader that the *property* of being $L(n)$ has nothing to do with the *space* $\mathcal{L}(n)$. We also note that being E_n does not imply the space is $L(n)$.

Remark 2.4. Note that our definition of a $L(n)$ space is different from that of a Q_n space used in [**4**]. Kraines and Lada restrict to the case when X is connected, in which case L could be replaced with $Q = \Omega^\infty\Sigma^\infty$. In this respect, our definition is more general.

We illustrate our definition with a sequence of examples (for more detailed exposition and proofs see [**4**] or [**2**, V]).

Example 2.5.

(1) By definition, every based space is a $L(0)$ space.
(2) A based space X is $L(1)$, if the canonical map $X \to LX$ admits a retraction μ_X, which we can regard as the multiplication on X.

[1] It is helpful to think of this basepoint as the multiplicative unit.

(3) A space X is $L(2)$, if it is $L(1)$ and we have a specified homotopy $I \times L^2X \to X$ between $\mu_X\mu$ and $\mu(\mu)$. In other words, X is an L-algebra in the homotopy category of pointed spaces.
(4) In the case L is the monad associated to an operad, by a result of Lada [**2**], X is $L(\infty)$ if and only if it admits an L-algebra structure. If the components of X form a group under the induced multiplication, then X is $L(\infty)$ if and only if it has the homotopy type of an infinite loop space.

There is an obvious analogue of the above definition with based spaces replaced by spectra under S^0, where L is the monad whose algebras are E_∞ ring spectra in this category [**6**, 6.2]. So we obtain an analogous categorical filtration between spectra under S^0 and E_∞ ring spectra.

Applying this equivalence to the definition of $L(n)$ spectra, we see that the definition of an $L(2)$ spectrum is precisely the definition of a H_∞ ring spectrum under S^0 [**1**]. By Lemma 2.1 we see that applying Σ^∞_+ to the map

$$KL \to BU$$

of $L(2)$ spaces constructed by Kraines and Lada we obtain a map of H_∞ ring spectra augmented over S^0.

To see that $\Sigma^\infty_+ KL$ is not an E_∞ ring spectrum we apply the argument of [**4**, §8]. There they show that the Postnikov system for KL gives rise to a fibration sequence:

$$KL_{\leq 29} \to KL_{\leq 28} \simeq BU_{\leq 28} \xrightarrow{\tau} K(\mathbb{Z}/(4 \cdot 15!), 30).$$

If KL were an infinite loop space, $KL_{\leq 29}$ would be as well and the k-invariant would be an infinite loop map. However they demonstrate that τ can not be delooped twice to a multiplicative map and so the above Postnikov fibration can not be delooped twice to a Postnikov system of A_∞ spaces. As a consequence $KL_{\leq 29}$ and KL fail to be E_3 spaces and the corresponding suspension spectra fail to have induced E_3 structures.

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